An Introduction to System-theoretic Methods for Model Reduction - Part III
Preserving System Structure

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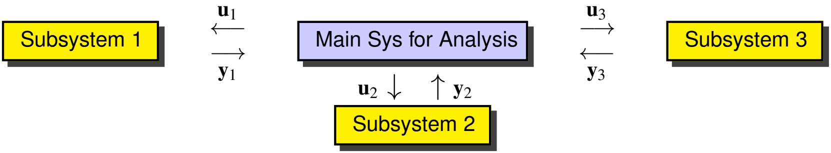
Outline for Second Lecture

Implicitly Structured Dynamical Systems
Energy-based Modeling and Reduction

- Basic energy-based modeling notions for dynamical systems.
  - Supply rates and dissipativity for linear dynamical systems
  - Storage functions and Linear Matrix Inequalities
  - Preserving dissipativity with interpolatory model reduction
- Structurally passive nonlinear dynamical systems: port-Hamiltonian systems
  - Ensemble-based methods (POD)
  - Ensemble-free methods built on interpolatory methods

Goals of Model Reduction



- Replace high-order complex subsystems with low-order, (but high-fidelity) surrogates. Encode high resolution/fine grain structure of the subsystem response acquired offline into compact, efficient online surrogates.
- Avoid using (expensive) human resources. Want the process to be (relatively) automatic and capable of producing reliable high-fidelity surrogates.
- Should respect underlying "physics" High fidelity may not be enough - surrogate models should behave "physically" and respect underlying conservation laws.

Energy-based modeling of dynamical systems

DynSys: $\mathbf{u}(t) \in \mathbb{U} \longrightarrow$ $\boxed{\begin{array}{ll} \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}(t) & \\ & \mathbf{x}(t) \in \mathbb{X} \\ \mathbf{y}(t) = \mathbf{C}\,\mathbf{x} & \end{array}}$ $\longrightarrow \mathbf{y}(t) \in \mathbb{Y}$

- Assume: linear, time-invariant, asymp stable, min sys realization.

Energy-based modeling: allows for the system to extract, store, and return value ("energy") to/from the environment.
(inspired by: "Gibbs free energy", "available work", "karma" ...)

Key Modeling Element:

- *Energy/Value Supply Rate*, $w\colon \mathbb{Y} \times \mathbb{U} \to \mathbb{R}$ with $w(\mathbf{y}(\cdot), \mathbf{u}(\cdot)) \in \mathcal{L}^1_{loc}$
  $w(\mathbf{y}(t), \mathbf{u}(t))$ models the instantaneous exchange of value/energy of the system with the environment via inputs and outputs.

Supply rates and dissipativity

Examples of supply rates:

- $w(\mathbf{y}(t), \mathbf{u}(t)) = \mathbf{u}(t)^T \mathbf{y}(t)$ (work ⇒ “Passive systems”)
- $w(\mathbf{y}(t), \mathbf{u}(t)) = \frac{1}{2}(\|\mathbf{u}(t)\|^2 - \|\mathbf{y}(t)\|^2)$ (instantaneous gain ⇒ “Contractive systems”)
- $w(\mathbf{y}(t), \mathbf{u}(t)) = \frac{1}{2}\begin{pmatrix} \mathbf{y}(t) & \mathbf{u}(t) \end{pmatrix} \begin{bmatrix} -\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M} \end{bmatrix} \begin{pmatrix} \mathbf{y}(t) \\ \mathbf{u}(t) \end{pmatrix}$

  with $\mathbf{M} \geq 0$ $\mathbf{N} \geq 0$ (General quadratic supply rate)

For a given energy/value supply rate, $w(\mathbf{y}(\cdot), \mathbf{u}(\cdot))$, a dynamical system is ***dissipative with respect to*** w, if whenever the system starts in an equilibrium state at $t_0 = 0$,

$$\int_0^t w(\mathbf{y}(t), \mathbf{u}(t))\, dt \geq 0 \qquad \text{for all } t \geq 0$$

Starting from equilibrium, a dissipative system can never lose more energy to the environment than it has gained.

Dissipative systems can store energy (but maybe not give it back)

A *storage function* associated with the supply rate, w, is a scalar-valued function of state, $\mathrm{H} : \mathbb{X} \to \mathbb{R}^+$, that satisfies for any $0 \leq t_0 < t_1$

$$\mathrm{H}(\mathbf{x}(t_1)) - \mathrm{H}(\mathbf{x}(t_0)) \leq \int_{t_0}^{t_1} w(\mathbf{y}(t), \mathbf{u}(t))\, dt \quad \text{(dissipation inequality)}$$

- $\mathrm{H}(\mathbf{x})$ is a measure of "internal energy" in the system when it in state $\mathbf{x}$.
- The dissipation inequality asserts the change in internal energy cannot exceed the net energy absorbed or delivered by the system from/to the environment.
- Dissipative systems cannot create "energy" internally apart from what is delivered from the environment.

Available Storage – max energy extractable from a system state

For any *storage function* $\mathsf{H} : \mathbb{X} \to \mathbb{R}^+$ associated with the supply rate, w: for any $0 \le t_0 < t_1$, $\mathbf{u} \in \mathbb{U}$

$$\mathsf{H}(\mathbf{x}(t_1)) - \mathsf{H}(\mathbf{x}(t_0)) \le \int_{t_0}^{t_1} w(\mathbf{y}(t), \mathbf{u}(t))\, dt \quad \text{(dissipation inequality)}$$

- Starting with $\mathbf{x}(0) = \hat{\mathbf{x}}$, then for any $\mathbf{u} \in \mathbb{U}$

$$-\mathsf{H}(\hat{\mathbf{x}}) \le \mathsf{H}(\mathbf{x}(\tau)) - \mathsf{H}(\hat{\mathbf{x}}) \le \int_0^\tau w(\mathbf{y}(t), \mathbf{u}(t))\, dt \implies -\int_0^\tau w(\mathbf{y}(t), \mathbf{u}(t))\, dt \le \mathsf{H}(\hat{\mathbf{x}})$$

$$\implies \sup_{\tau > 0, \mathbf{u} \in \mathbb{U}} \left(-\int_0^\tau w(\mathbf{y}(t), \mathbf{u}(t))\, dt \right) \le \mathsf{H}(\hat{\mathbf{x}})$$

- $\mathsf{H}(\mathbf{x}) \ge \mathsf{H}(\mathbf{0})$ and wlog we can assume $\mathsf{H}(\mathbf{0}) = 0$

"Available Storage"

$$\mathsf{H}_{min}(\hat{\mathbf{x}}) = \sup_{\substack{\mathbf{u} \in \mathbb{U} \\ \tau > 0}} \left\{ -\int_0^\tau w(\mathbf{y}(t), \mathbf{u}(t))\, dt \quad \middle| \quad \mathbf{x}(0) = \hat{\mathbf{x}} \right\}$$

- This is the maximum amount of energy that the system can make available to do work on the environment, starting at the initial state $\hat{\mathbf{x}}$. For any $\hat{\mathbf{x}} \in \mathbb{X}$, $\mathsf{H}_{min}(\hat{\mathbf{x}}) \le \mathsf{H}(\hat{\mathbf{x}})$.

Required Supply : min energy required to set a system state

For any *storage function* $\mathsf{H} : \mathbb{X} \to \mathbb{R}^+$ associated with the supply rate, w: for any $0 \leq t_0 < t_1$, $\mathbf{u} \in \mathbb{U}$

$$\mathsf{H}(\mathbf{x}(t_1)) - \mathsf{H}(\mathbf{x}(t_0)) \leq \int_{t_0}^{t_1} w(\mathbf{y}(t), \mathbf{u}(t))\, dt \quad \text{(dissipation inequality)}$$

- If $\mathbf{u} \in \mathbb{U}$ steers the system from a null initial state $\mathbf{x}(0) = \mathbf{0}$ to a final state $\mathbf{x}(\tau) = \hat{\mathbf{x}}$ at $t = \tau$, then

$$\mathsf{H}(\hat{\mathbf{x}}) = \mathsf{H}(\hat{\mathbf{x}}) - \mathsf{H}(\mathbf{0}) \leq \int_0^{\tau} w(\mathbf{y}(t), \mathbf{u}(t))\, dt$$

"Required Supply"

$$\mathsf{H}_{max}(\hat{\mathbf{x}}) = \inf_{\substack{\tau \geq 0 \\ \mathbf{u} \in \mathbb{U}}} \left\{ \int_0^{\tau} w(\mathbf{y}(t), \mathbf{u}(t))\, dt \;\middle|\; \begin{array}{l} \mathbf{x}(0) = \mathbf{0} \\ \mathbf{x}(\tau) = \hat{\mathbf{x}} \end{array} \right\}$$

- This is the minimum amount of energy that the system must absorb from the environment to move from a null state to a final state of $\hat{\mathbf{x}}$.
  For any $\hat{\mathbf{x}} \in \mathbb{X}$, $\mathsf{H}(\hat{\mathbf{x}}) \leq \mathsf{H}_{max}(\hat{\mathbf{x}})$.

- Dissipativity is an exogenous system property externally characterized; **dependent** on supply rate but **independent** of system realization.
- Storage functions are endogenous to a system internally characterized; **dependent** both on supply rate and system realization.
- For dissipative systems, both $\mathsf{H}_{min}(\mathbf{x})$ (available storage) and $\mathsf{H}_{max}(\mathbf{x})$ (required supply) are valid storage functions.

Quadratic supply rates imply quadratic storage functions

$$\mathsf{H}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \quad \text{for some } 0 < \mathbf{Q}$$

$$\mathsf{H}_{min}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}_{min}\mathbf{x} \qquad \mathsf{H}_{max}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}_{max}\mathbf{x}$$

$$\text{and} \quad 0 < \mathbf{Q}_{min} \leq \mathbf{Q} \leq \mathbf{Q}_{max}$$

State-space conditions for dissipativity

Take the supply rate to be a general quadratic:

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \frac{1}{2}\begin{pmatrix} \mathbf{y}^T & \mathbf{u}^T \end{pmatrix}\begin{bmatrix} -\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M} \end{bmatrix}\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix} \text{ with } \mathbf{M} \geq 0, \mathbf{N} \geq 0 \text{ and}$$

suppose $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \text{ for } \mathbf{Q} > 0.$$

- The dissipation inequality implies

$$\frac{d}{dt}\mathsf{H}(\mathbf{x}(t)) \leq w(\mathbf{y}(t),\mathbf{u}(t)).$$

$$\implies \mathbf{x}^T\mathbf{Q}\dot{\mathbf{x}} = \mathbf{x}^T\mathbf{Q}(\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}) \leq \frac{1}{2}(\mathbf{x}^T\mathbf{C}^T\ \mathbf{u}^T)\begin{bmatrix} -\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M} \end{bmatrix}\begin{pmatrix} \mathbf{C}\mathbf{x} \\ \mathbf{u} \end{pmatrix}$$

$$\implies \frac{1}{2}\mathbf{x}^T(\mathbf{Q}\mathbf{A}+\mathbf{A}^T\mathbf{Q}+\mathbf{C}^T\mathbf{N}\mathbf{C})\mathbf{x} + \mathbf{x}^T(\mathbf{Q}\mathbf{B}-\mathbf{C}^T\Omega)\mathbf{u} - \frac{1}{2}\mathbf{u}^T\mathbf{M}\mathbf{u} \leq 0$$

- The system is dissipative wrt the supply w if and only if the LMI

$$\begin{bmatrix} \mathbf{Q}\,\mathbf{A}+\mathbf{A}^T\mathbf{Q}+\mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{Q}\,\mathbf{B}-\mathbf{C}^T\Omega \\ \mathbf{B}^T\mathbf{Q}-\Omega^T\mathbf{C} & -\mathbf{M} \end{bmatrix} \leq 0$$

has a positive-definite solution matrix, $\mathbf{Q} > 0$.

Special case: Passive systems

Take the supply rate to be:

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \mathbf{u}(t)^T\mathbf{y}(t) = \frac{1}{2}\begin{pmatrix} \mathbf{y}^T & \mathbf{u}^T \end{pmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}$$

(defining $\mathbf{M} = \mathbf{N} = 0$ and $\Omega = \mathbf{I}$)

and suppose $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \ \text{ for } \ \mathbf{Q} > 0.$$

- The system is **passive** with the storage function $\mathsf{H}(\mathbf{x})$, if and only if $\mathbf{Q}$ is a positive-definite solution to the LMI:

$$\begin{bmatrix} \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T\mathbf{Q} & \mathbf{Q}\,\mathbf{B} - \mathbf{C}^T \\ \mathbf{B}^T\mathbf{Q} - \mathbf{C} & 0 \end{bmatrix} \leq 0 \Leftrightarrow \begin{array}{c} \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T\mathbf{Q} \leq 0 \\ \mathbf{Q}\,\mathbf{B} = \mathbf{C}^T \end{array} \quad \text{(Luré LMI)}$$

- Passive systems have **port-Hamiltonian** realizations. Take $\mathbf{Q}\,\mathbf{A} = \mathbf{J} - \mathbf{R}$ with $\mathbf{J} = -\mathbf{J}^T$ and $\mathbf{R} = \mathbf{R}^T$ (skew-symm + symm).

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \Leftrightarrow \mathbf{Q}\dot{\mathbf{x}} = \mathbf{Q}\mathbf{A}\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u} \Leftrightarrow \mathbf{Q}\,\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\mathbf{x} + \mathbf{C}^T\mathbf{u}$$

$$\mathbf{Q}\,\mathbf{A} + \mathbf{A}^T\mathbf{Q} = -2\mathbf{R} \leq 0 \quad \Leftrightarrow \quad \mathbf{R} \geq 0$$

Special case: γ-contractive systems

Pick $\gamma > 0$ and take the supply rate to be:

$$w(\mathbf{y}(t), \mathbf{u}(t)) = \frac{1}{2}\left(\gamma^2 \|\mathbf{u}(t)\|^2 - \|\mathbf{y}(t)\|^2\right) = \frac{1}{2}\begin{pmatrix} \mathbf{y}^T & \mathbf{u}^T \end{pmatrix} \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \gamma^2 \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix}$$

(defining $\mathbf{M} = \gamma^2 \mathbf{I}$, $\mathbf{N} = -\mathbf{I}$ and $\Omega = \mathbf{0}$)

and suppose $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{Q}\mathbf{x} \ \text{ for } \ \mathbf{Q} > 0.$$

- The system is γ-**contractive** with the storage function $\mathsf{H}(\mathbf{x})$, if and only if $\mathbf{Q}$ is a positive-definite solution to the LMI:

$$\begin{bmatrix} \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T \mathbf{Q} + \mathbf{C}^T \mathbf{C} & \mathbf{Q}\,\mathbf{B} \\ \mathbf{B}^T \mathbf{Q} & -\gamma^2 \mathbf{I} \end{bmatrix} \leq 0 \Leftrightarrow \quad \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T \mathbf{Q} + \mathbf{C}^T \mathbf{C} + \frac{1}{\gamma^2}\mathbf{Q}\,\mathbf{B}\mathbf{B}^T \mathbf{Q} \leq 0$$

(Riccati Matrix Inequality)

- If $\mathcal{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ is the transfer function for the system then the system is γ-contractive if and only if $\|\mathcal{G}\|_{\mathcal{H}_\infty} \leq \gamma$. This is an important property to insure when designing model-based stabilizing controllers that are robust to model uncertainty.

The way forward...

- Dissipative systems have realizations that encode energy flux constraints determined by the supply rate and the underlying dissipation framework via linear matrix inequalities (LMIs).
- Seek model reduction strategies that preserve this structure $\implies$ Create reduced order surrogate models that have high fidelity and respect original dissipation constraints (this is sensible because dissipation is an exogenous property).
- Warning ! Direct use of LMIs can be computationally untenable due to high model order. (Complexity can grow like $\mathcal{O}(n^4)$!)

Addressing this properly remains a topic of interest (and another talk...), but for the time being assume that a storage function, $\mathrm{H}(\mathbf{x})$, is known.

- $\mathrm{H}(\mathbf{x})$ is dependent on both the system realization and the supply rate.
- For linear time invariant systems with a quadratic supply rate, $\mathrm{H}(\mathbf{x})$ can be assumed to be quadratic without loss of generality.

Preserving dissipativity in reduced order models

Take the supply rate to be a general quadratic:

$$w(\mathbf{y}(t),\mathbf{u}(t)) = \frac{1}{2}\begin{pmatrix} \mathbf{y}^T & \mathbf{u}^T \end{pmatrix}\begin{bmatrix} -\mathbf{N} & \Omega \\ \Omega^T & \mathbf{M} \end{bmatrix}\begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix} \text{ with } \mathbf{M} \geq 0, \mathbf{N} \geq 0$$

and $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \;\text{ for }\; \mathbf{Q} > 0.$$

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}} \longrightarrow \boxed{\begin{aligned} \mathbf{Q}\dot{\mathbf{x}}_r &= (\mathbf{J}-\mathbf{R})\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}}$$

(Original Realization) (Dissipative Realization)

- $\mathbf{Q}\,\mathbf{A} = \mathbf{J} - \mathbf{R}$ with $\mathbf{J} = -\mathbf{J}^T$ and $\mathbf{R} = \mathbf{R}^T$ (skew-symm + symm).
- “Project dynamics” by approximating $\mathbf{x}(t) \approx \mathbf{V}_r\mathbf{x}_r(t)$:

$$\mathbf{V}_r^T\mathbf{Q}\left(\mathbf{V}_r\dot{\mathbf{x}}_r(t) - \mathbf{A}\mathbf{V}_r\mathbf{x}_r(t) - \mathbf{B}\mathbf{u}(t)\right) = 0 \quad \text{(Petrov-Galerkin)}$$

or equivalently,

$$\mathbf{V}_r^T\left(\mathbf{Q}\mathbf{V}_r\dot{\mathbf{x}}_r(t) - (\mathbf{J}-\mathbf{R})\mathbf{V}_r\mathbf{x}_r(t) - \mathbf{Q}\mathbf{B}\mathbf{u}(t)\right) = 0 \quad \text{(Ritz-Galerkin)}$$

for some choice of subspace $\mathcal{V}_r = \mathrm{Ran}(\mathbf{V}_r)$.

Preserving dissipativity in reduced order models

Take the supply rate to be a general quadratic:

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and $\mathsf{H}(\mathbf{x})$ is an associated quadratic storage function:

$$\mathsf{H}(\mathbf{x}) = \frac{1}{2}\,\mathbf{x}^T\mathbf{Q}\mathbf{x} \;\text{ for }\; \mathbf{Q} > 0.$$

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}} \quad \longrightarrow \quad \boxed{\begin{aligned} \mathbf{Q}\dot{\mathbf{x}}_r &= (\mathbf{J}-\mathbf{R})\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}}$$

(Original Realization) (Dissipative Realization)

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or equivalently,

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for some choice of subspace $\mathcal{V}_r = \mathrm{Ran}(\mathbf{V}_r)$.

Preserving dissipativity in reduced order models

$$\boxed{\begin{aligned}\mathbf{Q}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x}\end{aligned}} \longrightarrow \boxed{\begin{aligned}\mathbf{Q}_r\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\mathbf{x}_r + \mathbf{Q}_r\mathbf{B}_r\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r\end{aligned}}$$

(Dissipative realization) (Reduced dissipative model)

- $\mathbf{V}_r^T\left(\mathbf{Q}\mathbf{V}_r\dot{\mathbf{x}}_r(t) - (\mathbf{J}-\mathbf{R})\mathbf{V}_r\mathbf{x}_r(t) - \mathbf{Q}\mathbf{B}\mathbf{u}(t)\right) = 0$ (Ritz-Galerkin)

  for some choice of subspace $\mathcal{V}_r = \mathrm{Ran}(\mathbf{V}_r)$.

- Leads to a reduced model defined by

$$\mathbf{Q}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r, \quad \mathbf{J}_r = \mathbf{V}_r^T\mathbf{J}\mathbf{V}_r, \quad \mathbf{R}_r = \mathbf{V}_r^T\mathbf{R}\mathbf{V}_r,$$

$$\mathbf{C}_r = \mathbf{C}\mathbf{V}_r, \quad \mathbf{B}_r = \mathbf{Q}_r^{-1}\mathbf{V}_r^T\mathbf{Q}\mathbf{B}$$

Is this reduced model dissipative with respect to the same supply rate ?

Preserving dissipativity in reduced order models

The reduced model is defined by

$$\mathbf{Q}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r, \quad \mathbf{J}_r = \mathbf{V}_r^T\mathbf{J}\mathbf{V}_r, \quad \mathbf{R}_r = \mathbf{V}_r^T\mathbf{R}\mathbf{V}_r, \quad \mathbf{C}_r = \mathbf{C}\mathbf{V}_r, \quad \mathbf{B}_r = \mathbf{Q}_r^{-1}\mathbf{V}_r^T\mathbf{Q}\mathbf{B}$$

- Evidently, $\mathbf{Q}_r > 0$, $\mathbf{J}_r = -\mathbf{J}_r^T$ and $\mathbf{R}_r = \mathbf{R}_r^T$.

The original storage, $\mathbf{Q} > 0$, solves

$$\begin{bmatrix} \mathbf{Q}\,\mathbf{A} + \mathbf{A}^T\,\mathbf{Q} + \mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{Q}\,\mathbf{B} - \mathbf{C}^T\Omega \\ \mathbf{B}^T\,\mathbf{Q} - \Omega^T\mathbf{C} & -\mathbf{M} \end{bmatrix} = \begin{bmatrix} -2\mathbf{R} + \mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{Q}\,\mathbf{B} - \mathbf{C}^T\Omega \\ \mathbf{B}^T\,\mathbf{Q} - \Omega^T\mathbf{C} & -\mathbf{M} \end{bmatrix} \leq 0$$

$$\Longrightarrow \begin{bmatrix} \mathbf{Q}_r\,\mathbf{A}_r + \mathbf{A}_r{}^T\,\mathbf{Q}_r + \mathbf{C}_r^T\mathbf{N}\mathbf{C}_r & \mathbf{Q}_r\,\mathbf{B}_r - \mathbf{C}_r^T\Omega \\ \mathbf{B}_r^T\,\mathbf{Q}_r - \Omega^T\mathbf{C}_r & -\mathbf{M} \end{bmatrix} = \begin{bmatrix} -2\mathbf{R}_r + \mathbf{C}_r^T\mathbf{N}\mathbf{C}_r & \mathbf{Q}_r\,\mathbf{B}_r - \mathbf{C}_r^T\Omega \\ \mathbf{B}_r^T\,\mathbf{Q}_r - \Omega^T\mathbf{C}_r & -\mathbf{M} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{V}_r^T & 0 \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} -2\mathbf{R} + \mathbf{C}^T\mathbf{N}\mathbf{C} & \mathbf{Q}\,\mathbf{B} - \mathbf{C}^T\Omega \\ \mathbf{B}^T\,\mathbf{Q} - \Omega^T\mathbf{C} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{V}_r & 0 \\ 0 & \mathbf{I} \end{bmatrix} \leq 0$$

- Thus, $\mathbf{R}_r \geq 0$ and $\mathbf{A}_r = \mathbf{Q}_r^{-1}(\mathbf{J}_r - \mathbf{R}_r)$ is asymp stable.
- $\Rightarrow$ The reduced system will be dissipative for *any* choice of $\mathcal{V}_r$

Finding effective reduced order dissipative models

$$\boxed{\begin{aligned} \mathbf{Q}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{x} + \mathbf{Q}\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{x} \end{aligned}} \longrightarrow \boxed{\begin{aligned} \mathbf{Q}_r\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\mathbf{x}_r + \widehat{\mathbf{B}}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r \end{aligned}}$$

(Original dissipative realization) (Reduced dissipative realization)

- Fourier Transforms: $\mathbf{u}(t) \mapsto \hat{\mathbf{u}}(\omega), \quad \mathbf{y}(t) \mapsto \hat{\mathbf{y}}(\omega)$

  Original response: $\hat{\mathbf{y}}(\omega) = \mathcal{G}(\imath\omega)\hat{\mathbf{u}}(\omega)$

  Reduced response: $\hat{\mathbf{y}}_r(\omega) = \mathcal{G}_r(\imath\omega)\hat{\mathbf{u}}(\omega)$

  with transfer functions:

$\mathcal{G}(s) = \mathbf{C}(s\mathbf{Q}-(\mathbf{J}-\mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}$ and $\mathcal{G}_r(s) = \mathbf{C}_r(s\mathbf{Q}_r-(\mathbf{J}_r-\mathbf{R}_r))^{-1}\widehat{\mathbf{B}}_r$.

- $$\hat{\mathbf{y}}(\omega) - \hat{\mathbf{y}}_r(\omega) = \Big(\mathcal{G}(\imath\omega) - \mathcal{G}_r(\imath\omega)\Big)\,\hat{\mathbf{u}}(\omega)$$

Find a modeling space $\mathcal{V}_r$ so that $\mathcal{G}_r(\imath\omega) \approx \mathcal{G}(\imath\omega)$ for $\omega \in \mathbb{R}$.

Finding effective reduced order dissipative models

$$\boxed{\mathcal{G}(s) = \mathbf{C}(s\mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}} \longrightarrow \boxed{\mathcal{G}_r(s) = \mathbf{C}_r(s\mathbf{Q}_r - (\mathbf{J}_r - \mathbf{R}_r))^{-1}\mathbf{Q}_r\mathbf{B}_r}$$

(Original system) (Reduced system)

Find a reduction space $\mathcal{V}_r$ so that $\mathcal{G}_r(\imath\omega) \approx \mathcal{G}(\imath\omega)$ for $\omega \in \mathbb{R}$ and $\mathcal{G}_r(s)$ is dissipative wrt same supply rate as $\mathcal{G}$.

Heuristic: $\mathcal{G}_r$ will be a best (rational) approximation to $\mathcal{G}$ when $|\mathcal{G}(\imath\omega) - \mathcal{G}_r(\imath\omega)| \approx$ constant for $\omega \in \mathbb{R}$ (SISO case).

$\Longrightarrow$ Suggests that symmetric distribution of poles and zeros of $|\mathcal{G}(s) - \mathcal{G}_r(s)|$ will be optimal. $\Rightarrow$ **Interpolate !**

- **zeros** are points of interpolation,
  poles include the poles of $\mathcal{G}_r$ which are computatable
- Iteratively rebalance pole/zero distribution by forming dissipative interpolants. Similar process for $\mathcal{H}_2/\mathcal{H}_\infty$-quasioptimal schemes for port-Hamiltonian reduction "PH-IRKA". (more on this later...)
- MIMO case is similar but interpolation occurs only in tangent directions given by (vector) residues of $\mathcal{G}_r(s)$.

Interpolation by reduced order dissipative systems

Construct a modeling subspace $\mathcal{V}_r$ that forces interpolation.

Interpolatory projections that preserve dissipativity

Given interpolation points $\sigma_1, ..., \sigma_r$ and

tangent directions $\mathbb{b}_1, ..., \mathbb{b}_r$, construct

$$\mathbf{V}_r = [(\sigma_1 \mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_1, \ldots, (\sigma_r \mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_r].$$

Then with $\mathbf{Q}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r, \quad \mathbf{J}_r = \mathbf{V}_r^T\mathbf{J}\mathbf{V}_r, \quad \mathbf{R}_r = \mathbf{V}_r^T\mathbf{R}\mathbf{V}_r, \quad \mathbf{C}_r = \mathbf{C}\mathbf{V}_r, \quad \mathbf{Q}_r\mathbf{B}_r = \mathbf{V}_r^T\mathbf{Q}\mathbf{B}$

the reduced model, $\mathcal{G}_r$: $\begin{cases} \mathbf{Q}_r\dot{\mathbf{x}}_r = (\mathbf{J}_r - \mathbf{R}_r)\mathbf{x}_r + \mathbf{Q}_r\mathbf{B}_r\,\mathbf{u}, \\ \mathbf{y}_r = \mathbf{C}_r\mathbf{x}_r \end{cases}$

is stable, minimal, dissipative wrt the given supply rate, w,

and $\quad \mathcal{G}_r(\sigma_i)\mathbb{b}_i = \mathcal{G}(\sigma_i)\mathbb{b}_i \quad$ for $\; i = 1, ..., r$.

How to choose interpolation points ?

- $\Phi(s) = \log |\mathcal{G}(s) - \mathcal{G}_r(s)|$ is a potential function
  · has positive singularities at system eigenvalues.
  · has negative singularities at interpolation points.
  · is harmonic everywhere else - electrostatic analogy
- Locate interpolation points (negative point charges) to balance equipotentials of $\log |\mathcal{G}(s) - \mathcal{G}_r(s)|$ (makes $\log |\mathcal{G}(s) - \mathcal{G}_r(s)|$ nearly constant along the imaginary axis)
- Interpolate at points that mirror singularities across the imaginary axis (but there are too many !)
- So mirror "equivalent charges" instead; e.g., Ritz values. (which are the poles of the reduced dissipative model, $\mathcal{G}_r$).

(Near) Best Dissipative Reduced Approximation

$$\boxed{\mathcal{G}(s) = \mathbf{C}(s\mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}} \longrightarrow \boxed{\mathcal{H}_r(s) = \mathbf{C}_r(s\mathbf{Q}_r - (\mathbf{J}_r - \mathbf{R}_r))^{-1}\mathbf{Q}_r\mathbf{B}_r}$$

(Original system) (Reduced system)

- Force $\mathcal{G}(-\widehat{\lambda}_k) = \mathcal{G}_r(-\widehat{\lambda}_k)$ at reduced system poles: $\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_r$.
- By choosing an subspace $\mathcal{V}_r$ that forces symmetric interpolation, we expect
  $\mathcal{G}_r(\imath\omega) \approx \mathcal{G}(\imath\omega)$ for $\omega \in \mathbb{R}$ and also $\mathcal{G}_r(s)$ is a dissipative system with respect to the same supply rate.
- MIMO case: if $\mathcal{G}_r(s) = \displaystyle\sum_{k=1}^{r} \frac{\mathbb{c}_k \mathbb{b}_k^T}{s - \widehat{\lambda}_k}$ then force $\mathcal{G}(-\widehat{\lambda}_k)\mathbb{b}_k = \mathcal{G}_r(-\widehat{\lambda}_k)\mathbb{b}_k$.

  (also a necc condition for $\mathcal{H}_2$-optimality)

Dissipation-preserving Model Reduction

Iterative correction to force interpolation at reflected reduced order poles: $\mathcal{G}_r(-\widehat{\lambda}_k)\mathbb{b}_k = \mathcal{G}(-\widehat{\lambda}_k)\mathbb{b}_k$ for $k = 1, \ldots, r$

Algorithm ($\mathcal{H}_\infty/\mathcal{H}_2$-based MOR for dissipative systems)

1. *Make an initial shift selection* $\{\sigma_i\}_1^r$, *and tangent directions* $\{\mathbb{b}_i\}_1^r$.
2. *while (not converged)*
   1. $\mathbf{V}_r = [(\sigma_1\mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_1, \ldots, (\sigma_r\mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_r]$.
   2. *Set* $\widehat{\mathbf{V}_r} = \mathbf{V}_r\mathbf{L}^{-1}$ *with* $\mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r = \mathbf{L}^T\mathbf{L}$ *(so* $\mathbf{Q}_r = \widehat{\mathbf{V}_r}^T\mathbf{Q}\widehat{\mathbf{V}_r} = \mathbf{I}_r$*).*
   3. *Set* $\mathbf{J}_r = \widehat{\mathbf{V}}_r^T\mathbf{J}\widehat{\mathbf{V}}_r$, $\mathbf{R}_r = \widehat{\mathbf{V}}_r^T\mathbf{R}\widehat{\mathbf{V}}_r$, *and* $\mathbf{B}_r = \widehat{\mathbf{V}}_r^T\mathbf{Q}\mathbf{B}$.
   4. *Calculate left eigenvectors:* $\mathbf{w}_i^T(\mathbf{J}_r - \mathbf{R}_r) = \widehat{\lambda}_i\mathbf{w}_i^T$.
   5. *Set* $\sigma_i \longleftarrow -\widehat{\lambda}_i$ *and* $\mathbb{b}_i \longleftarrow \mathbf{B}_r^T\mathbf{w}_i$ *for* $i = 1, \ldots, r$
3. *Calculate final reduced dissipative system:*
   Find $\mathbf{V}_r = [(\sigma_1\mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_1, \ldots, (\sigma_r\mathbf{Q} - (\mathbf{J} - \mathbf{R}))^{-1}\mathbf{Q}\mathbf{B}\mathbb{b}_r]$.
   Set $\widehat{\mathbf{V}_r} = \mathbf{V}_r\mathbf{L}^{-1}$ *with* $\mathbf{V}_r^T\mathbf{Q}\mathbf{V}_r = \mathbf{L}^T\mathbf{L}$, *and* $\widehat{\mathbf{W}}_r = \mathbf{Q}\widehat{\mathbf{V}_r}$.
   Set $\mathbf{J}_r = \widehat{\mathbf{V}}_r^T\mathbf{J}\widehat{\mathbf{V}}_r$, $\mathbf{R}_r = \widehat{\mathbf{V}}_r^T\mathbf{R}\widehat{\mathbf{V}}_r$, $\mathbf{B}_r = \widehat{\mathbf{V}}_r^T\mathbf{Q}\mathbf{B}$, *and* $\mathbf{Q}_r = \mathbf{I}_r$.

(Gugercin, Polyuga, B, and van der Schaft, 2010) for passivity-preserving case

Extension to nonlinear systems (passive case)

Linear: $\boxed{\begin{aligned} \mathbf{Q}\dot{\mathbf{z}} &= (\mathbf{J}-\mathbf{R})\mathbf{z} + \mathbf{C}^T\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{z} \end{aligned}}$ with $\mathbf{Q} > \mathbf{0}$, $\mathbf{J} = -\mathbf{J}^T$, and $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$.

Nonlinear case: $\boxed{\begin{aligned} [\nabla^2\mathcal{E}(\mathbf{z})]\cdot\dot{\mathbf{z}} &= (\mathbf{J}-\mathbf{R})\mathbf{z} + \mathbf{C}^T\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{z} \end{aligned}}$

with $\mathcal{E}(\mathbf{z})$ uniformly convex, $\mathbf{J} = -\mathbf{J}^T$, and $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$.
$\mathbf{J}$, $\mathbf{R}$, and $\mathbf{C}$ could all depend on $\mathbf{z}$ as well.

Alternative (conjugate) representation:
Define $\mathbf{x} = \nabla\mathcal{E}(\mathbf{z})$ and $H(\mathbf{x}) = \sup_{\mathbf{z}} \left(\mathbf{x}^T\mathbf{z} - \mathcal{E}(\mathbf{z})\right)$. $\implies$ $\mathbf{z} = \nabla H(\mathbf{x})$.

Then $\boxed{\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}}H(\mathbf{x}) + \mathbf{C}^T\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\nabla_{\mathbf{x}}H(\mathbf{x}) \end{aligned}}$ $H(\mathbf{x})$ defines a *storage* function.

$H(\mathbf{x})$ is uniformly convex, $\mathbf{J} = -\mathbf{J}^T$, and $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$.
$\mathbf{J}$, $\mathbf{R}$, and $\mathbf{C}$ now all depend (potentially) on $\mathbf{x}$.

This is a “port-Hamiltonian” representation of the system.

Port-Hamiltonian (PH) Systems

Multi-Input/Multi-Output (MIMO) systems:

$$\mathbf{u}(t) \longrightarrow \boxed{\begin{array}{c} \dot{\mathbf{x}} = (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{C}^T\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\,\nabla_{\mathbf{x}} H(\mathbf{x}) \end{array}} \longrightarrow \mathbf{y}(t)$$

- $H : \mathbb{R}^n \to [0,\infty)$ is the *Hamiltonian*, defining the system internal energy as a function of instantaneous *state*, $\mathbf{x}(t)$.
- $\mathbf{J} = -\mathbf{J}^T$ is the *structure matrix* describing interconnection of energy storage components. (e.g., Kirchoff's Laws).
- $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$ is the *dissipation matrix* describing internal energy losses.
- Generalizes classical Hamiltonian systems: $\dot{\mathbf{x}} = \mathbf{J}\nabla_{\mathbf{x}} H(\mathbf{x})$.

Port-Hamiltonian (PH) Systems

$$\mathbf{u}(t) \longrightarrow \boxed{\begin{array}{c} \dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{C}^T \mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\, \nabla_{\mathbf{x}} H(\mathbf{x}) \end{array}} \longrightarrow \mathbf{y}(t)$$

Advantageous Features:

- PH systems are always *stable* and *passive*:

$$H(\mathbf{x}_1) - H(\mathbf{x}_0) \leq \int_{t_0}^{t_1} \mathbf{y}(t)^T \mathbf{u}(t)\, dt \quad (\Delta H \leq \text{total work}).$$

Why ?

Port-Hamiltonian (PH) Systems

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$$\frac{d}{dt} H(\mathbf{x}(t)) = \nabla_{\mathbf{x}} H(\mathbf{x})^T\, \dot{\mathbf{x}}$$

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Why ?

$$\frac{d}{dt} H(\mathbf{x}(t)) = \nabla_{\mathbf{x}} H(\mathbf{x})^T\, \dot{\mathbf{x}}$$

$$= \nabla_{\mathbf{x}} H(\mathbf{x})^T\, (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) \quad + \quad \nabla_{\mathbf{x}} H(\mathbf{x})^T\, \mathbf{C}^T \mathbf{u}(t)$$

Port-Hamiltonian (PH) Systems

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Why ?

$$\begin{aligned} \frac{d}{dt}H(\mathbf{x}(t)) &= \nabla_{\mathbf{x}}H(\mathbf{x})^T\,\dot{\mathbf{x}} \\ &= \nabla_{\mathbf{x}}H(\mathbf{x})^T\,(\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}}H(\mathbf{x}) \quad + \quad \nabla_{\mathbf{x}}H(\mathbf{x})^T\,\mathbf{C}^T\mathbf{u}(t) \\ &= \underbrace{-\nabla_{\mathbf{x}}H(\mathbf{x})^T\mathbf{R}\nabla_{\mathbf{x}}H(\mathbf{x})}_{\leq 0} \quad + \quad \underbrace{\mathbf{y}(t)^T\,\mathbf{u}(t)}_{\text{“power”}} \qquad \leq \mathbf{y}(t)^T\,\mathbf{u}(t) \end{aligned}$$

Port-Hamiltonian (PH) Systems

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- Closed under (power conserving) interconnection.

Port-Hamiltonian (PH) Systems

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- Closed under (power conserving) interconnection.

State space dimension, n, can be very large: $n \gg \dim \mathbf{u} = \dim \mathbf{y}$.
The input-output map $\mathbf{u} \mapsto \mathbf{y}$ is of primary interest.
“Internal state” $\mathbf{x}(t)$ is of secondary interest.

Goal: Reduce state space dimension without degrading input-output response; keep advantageous system features.

Maintain high fidelity and physical consistency (“structure”)

Finding a “smaller” PH system

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{J} - \mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{C}^T \mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\nabla_{\mathbf{x}} H(\mathbf{x}) \end{aligned} \quad \xrightarrow{?} \quad \begin{aligned} \dot{\mathbf{x}}_r &= (\mathbf{J}_r - \mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{C}_r^T \mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\,\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) \end{aligned}$$

(Original system) (Reduced system)

Want outputs to remain close, $\mathbf{y}_r(t) \approx \mathbf{y}(t)$, over a large class of possible inputs $\mathbf{u}(t)$.

Usual approach: Eliminate low value portions of state space.

Finding a “smaller” PH system

$$\boxed{\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{C}^T\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\nabla_{\mathbf{x}} H(\mathbf{x})\end{aligned}} \quad \stackrel{?}{\longrightarrow} \quad \boxed{\begin{aligned}\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{C}_r^T\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}_r\,\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r)\end{aligned}}$$

(Original system) (Reduced system)

Want outputs to remain close, $\mathbf{y}_r(t) \approx \mathbf{y}(t)$,
over a large class of possible inputs $\mathbf{u}(t)$.

Usual approach: Eliminate low value portions of state space.

Find subspaces $\mathcal{V}_r$, $\mathcal{W}_r$ such that

- $\mathbf{x}(t)$ stays close to $\mathcal{V}_r \implies \mathbf{x}(t) \approx \mathbf{V}_r\mathbf{x}_r(t)$
- $\nabla_{\mathbf{x}} H(\mathbf{x}(t))$ stays close to $\mathcal{W}_r \implies \nabla_{\mathbf{x}} H(\mathbf{x}(t)) \approx \mathbf{W}_r\mathbf{h}_r(t)$

...and neither $\mathcal{V}_r$ nor $\mathcal{W}_r$ depends on the input, $\mathbf{u}(t)$.

Finding a "smaller" PH system

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{B}^T\, \nabla_{\mathbf{x}} H(\mathbf{x}) \end{aligned}} \quad \stackrel{?}{\longrightarrow} \quad \boxed{\begin{aligned} \dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{B}_r^T\, \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) \end{aligned}}$$

(Original system) (Reduced system)

Assume that subspaces $\mathcal{V}_r$ and $\mathcal{W}_r$ have been found so that

$\mathbf{x}(t) \approx \mathbf{V}_r\mathbf{x}_r(t)$ and $\nabla_{\mathbf{x}} H(\mathbf{x}(t)) \approx \mathbf{W}_r\mathbf{h}_r(t)$.

How is a reduced PH system determined ?

Note that $\mathbf{W}_r\mathbf{h}_r(t) \approx \nabla_{\mathbf{x}} H(\mathbf{V}_r\mathbf{x}_r(t))$ implies

$$\mathbf{V}_r^T\mathbf{W}_r\mathbf{h}_r(t) \approx \mathbf{V}_r^T\nabla_{\mathbf{x}} H(\mathbf{V}_r\mathbf{x}_r(t)) = \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))$$

with a "reduced energy": $H_r(\mathbf{x}_r(t)) = H(\mathbf{V}_r\mathbf{x}_r(t))$

So, if biorthogonal bases for $\mathcal{V}_r$ and $\mathcal{W}_r$ are chosen (so $\mathbf{V}_r^T\mathbf{W}_r = \mathbf{I}$)

then $\mathbf{h}_r(t) \approx \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))$

Finding a "smaller" PH system

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{C}^T\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\nabla_{\mathbf{x}} H(\mathbf{x})\end{aligned}$$

(Original system)

$\xrightarrow{?}$

$$\begin{aligned}\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{C}_r^T\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}_r\,\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r)\end{aligned}$$

(Reduced system)

Substitute $\mathbf{x}(t) \approx \mathbf{V}_r\mathbf{x}_r(t)$ and $\nabla_{\mathbf{x}} H(\mathbf{V}_r\mathbf{x}_r(t)) \approx \mathbf{W}_r\mathbf{h}_r(t) \approx \mathbf{W}_r\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))$

$$\begin{aligned}\mathbf{V}_r\dot{\mathbf{x}}_r(t) &= (\mathbf{J}-\mathbf{R})\mathbf{W}_r\,\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t)) + \mathbf{C}^T\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}\,\mathbf{W}_r\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{x}}_r(t) &= (\mathbf{J}_r-\mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t)) + \mathbf{C}_r^T\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}_r\,\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))\end{aligned}$$

with $\mathbf{J}_r = \mathbf{W}_r^T\mathbf{J}\mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T\mathbf{R}\mathbf{W}_r$,

$\mathbf{C}_r = \mathbf{C}\mathbf{W}_r$, and $H_r(\mathbf{x}_r) = H(\mathbf{V}_r\mathbf{x}_r)$.

Finding a "smaller" PH system

$$\boxed{\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{C}^T\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\nabla_{\mathbf{x}} H(\mathbf{x})\end{aligned}} \quad \stackrel{?}{\longrightarrow} \quad \boxed{\begin{aligned}\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{C}_r^T\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}_r\,\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r)\end{aligned}}$$

(Original system) (Reduced system)

Substitute $\mathbf{x}(t) \approx \mathbf{V}_r\mathbf{x}_r(t)$ and $\nabla_{\mathbf{x}} H(\mathbf{V}_r\mathbf{x}_r(t)) \approx \mathbf{W}_r\mathbf{h}_r(t) \approx \mathbf{W}_r\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))$

$$\begin{aligned}\mathbf{W}_r^T\mathbf{V}_r\dot{\mathbf{x}}_r(t) &= \mathbf{W}_r^T(\mathbf{J}-\mathbf{R})\mathbf{W}_r\,\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t)) + \mathbf{W}_r^T\mathbf{C}^T\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}\,\mathbf{W}_r\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))\end{aligned}$$

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with $\mathbf{J}_r = \mathbf{W}_r^T\mathbf{J}\mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T\mathbf{R}\mathbf{W}_r$,

$\mathbf{C}_r = \mathbf{C}\mathbf{W}_r$, and $H_r(\mathbf{x}_r) = H(\mathbf{V}_r\mathbf{x}_r)$.

Finding a “smaller” PH system

$$\boxed{\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}}H(\mathbf{x}) + \mathbf{C}^T\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\,\nabla_{\mathbf{x}}H(\mathbf{x})\end{aligned}} \quad \overset{?}{\longrightarrow} \quad \boxed{\begin{aligned}\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\nabla_{\mathbf{x}_r}H_r(\mathbf{x}_r) + \mathbf{C}_r^T\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}_r\,\nabla_{\mathbf{x}_r}H_r(\mathbf{x}_r)\end{aligned}}$$

(Original system) (Reduced system)

Substitute $\mathbf{x}(t) \approx \mathbf{V}_r\mathbf{x}_r(t)$ and $\nabla_{\mathbf{x}}H(\mathbf{V}_r\mathbf{x}_r(t)) \approx \mathbf{W}_r\mathbf{h}_r(t) \approx \mathbf{W}_r\nabla_{\mathbf{x}_r}H_r(\mathbf{x}_r(t))$

$$\begin{aligned}\mathbf{W}_r^T\mathbf{V}_r\dot{\mathbf{x}}_r(t) &= \mathbf{W}_r^T(\mathbf{J}-\mathbf{R})\mathbf{W}_r\,\nabla_{\mathbf{x}_r}H_r(\mathbf{x}_r(t)) + \mathbf{W}_r^T\mathbf{C}^T\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}\,\mathbf{W}_r\nabla_{\mathbf{x}_r}H_r(\mathbf{x}_r(t))\end{aligned}$$

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with $\mathbf{J}_r = \mathbf{W}_r^T\mathbf{J}\mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T\mathbf{R}\mathbf{W}_r$,

$\mathbf{C}_r = \mathbf{C}\mathbf{W}_r$, and $H_r(\mathbf{x}_r) = H(\mathbf{V}_r\mathbf{x}_r)$.

POD for port-Hamiltonian systems (POD-PH)

Algorithm (POD-based MOR for port-Hamiltonian systems)

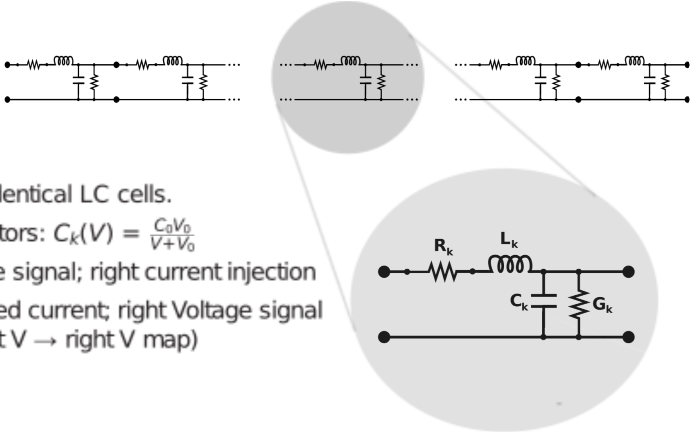
1. *Generate trajectory* $\mathbf{x}(t)$*, and collect snapshots:*
   $\mathbb{X} = [\mathbf{x}(t_0), \mathbf{x}(t_1), \mathbf{x}(t_2), \ldots, \mathbf{x}(t_N)]$.
2. *Truncate SVD of snapshot matrix,* $\mathbb{X}$*, to get POD basis,* $\widetilde{\mathbf{V}}_r$*, for the state space variables. Then* $\mathbf{x}(t) \approx \widetilde{\mathbf{V}}_r \tilde{\mathbf{x}}_r(t)$
3. *Collect associated force snapshots:*
   $\mathbb{F} = [\nabla_{\mathbf{x}} H(\mathbf{x}(t_0)), \nabla_{\mathbf{x}} H(\mathbf{x}(t_1)), \ldots, \nabla_{\mathbf{x}} H(\mathbf{x}(t_N))]$.
4. *Truncate SVD of* $\mathbb{F}$ *to get a second POD basis,* $\widetilde{\mathbf{W}}_r$*, spanning approximate range of* $\nabla_{\mathbf{x}} H(\mathbf{x}(t)) \approx \widetilde{\mathbf{W}}_r \tilde{\mathbf{h}}_r(t)$.
5. *Change bases* $\widetilde{\mathbf{W}}_r \mapsto \mathbf{W}_r$ *and* $\widetilde{\mathbf{V}}_r \mapsto \mathbf{V}_r$ *such that* $\mathbf{W}_r^T \mathbf{V}_r = \mathbf{I}$.

The POD-PH reduced system is

$$\dot{\mathbf{x}}_r = (\mathbf{J}_r - \mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{C}_r^T \mathbf{u}(t), \qquad \mathbf{y}_r(t) = \mathbf{C}_r \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r)$$

with $\mathbf{J}_r = \mathbf{W}_r^T \mathbf{J} \mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T \mathbf{R} \mathbf{W}_r$, $\mathbf{C}_r = \mathbf{C} \mathbf{W}_r$, and $H_r(\mathbf{x}_r) = H(\mathbf{V}_r \mathbf{x}_r)$.

Nonlinear Ladder Network Example



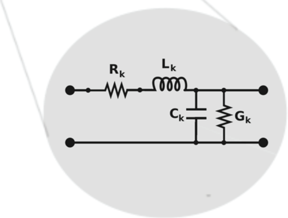
Network of 100 identical LC cells.

Nonlinear capacitors: $C_k(V) = \frac{C_0 V_0}{V + V_0}$

Input: left Voltage signal; right current injection

Output left induced current; right Voltage signal
(focus on left V $\rightarrow$ right V map)

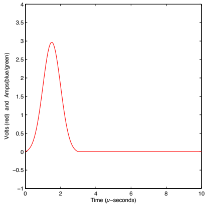
Nonlinear Ladder Network



Inductor - $\begin{cases} L_k = L_0 = 1\mu\text{H} \\ R_k = R_0 = 1\Omega \end{cases}$

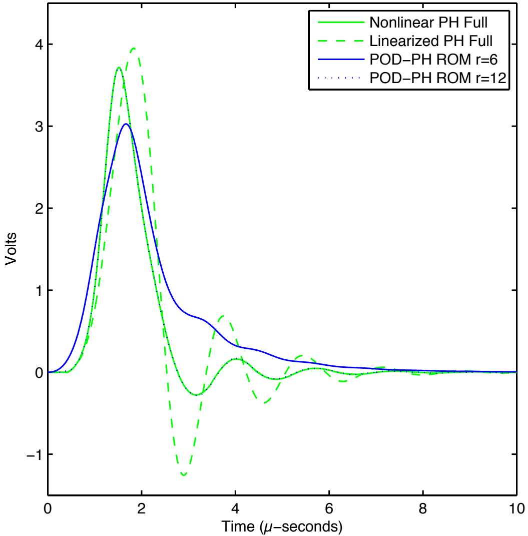
Capacitor - $\begin{cases} C_k(V) = \dfrac{C_0 V_0}{V + V_0} : & \begin{matrix} C_0 = 70pF \\ V_0 = 1.8V \end{matrix} \\ G_k = G_0 = 30\mu\mho \end{cases}$

Nonlinear Ladder Network



Input:
Gaussian pulse (3V pk)

ROM w/order r=12
accurate to 3.e-3



Augment the reduction subspaces

POD-PH provides an empirically driven choice for $\mathcal{V}_r$ and $\mathcal{W}_r$

... tied to an input ensemble

$\Rightarrow$ Only as good as the input ensembles chosen.

Other subspaces may be considered to replace/supplement POD:

- Find a choice of subspaces that is *asymptotically optimal* for small $\mathbf{u}$ (hence for small $\mathbf{x}$).

$\nabla_{\mathbf{x}} H(\mathbf{x}) \approx \mathbf{Q}^{-1}\mathbf{x}$ for a symmetric positive definite $\mathbf{Q} \in \mathbb{R}^{n\times n}$. (e.g., $\mathbf{Q} = \nabla^2 \mathcal{E}(\mathbf{0})$)

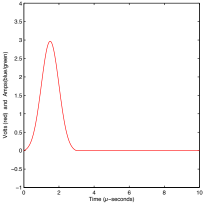
Leads to consideration of *Linear Port-Hamiltonian Systems*

$$\boxed{\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\mathbf{Q}^{-1}\mathbf{x} + \mathbf{C}^T\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\,\mathbf{Q}^{-1}\mathbf{x} \end{aligned}} \quad \longrightarrow \quad \boxed{\begin{aligned} \dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\mathbf{Q}_r^{-1}\mathbf{x}_r + \mathbf{C}_r^T\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{Q}_r^{-1}\mathbf{x}_r \end{aligned}}$$

(Original system) (Reduced system)

Find (sub)*optimal subspaces* for the linearized system; use them to augment the POD subspaces to reduce the original nonlinear system.

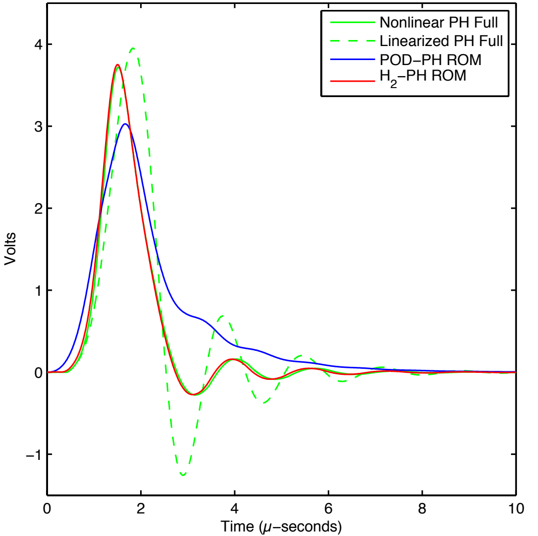
Ladder Network with POD/ensemble-free subspaces



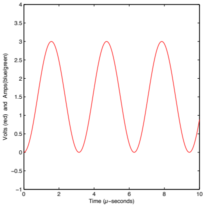
Input:
Gaussian pulse (3V pk)

POD-PH w/order r=6

$\mathcal{H}_\infty/\mathcal{H}_2$-PH w/order r=6
(roughly same accuracy as
POD-PH w/order r=12)



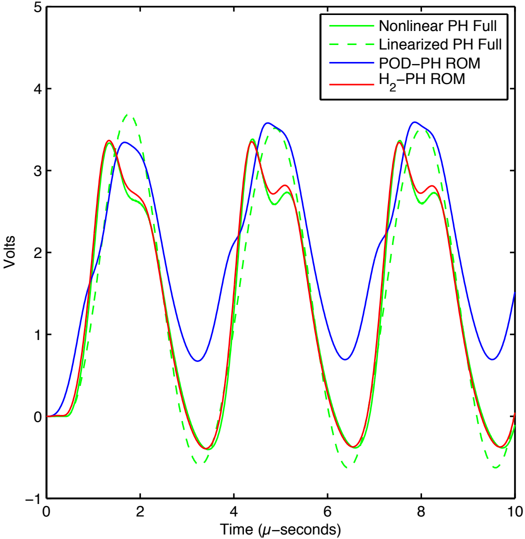
Ladder Network with POD/ensemble-free subspaces



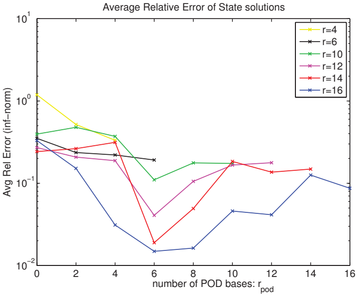
Input:
sinusoid (3V pk)

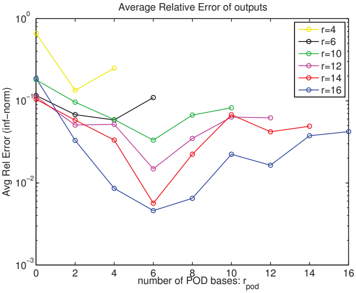
POD-PH w/order r=6

$\mathcal{H}_\infty/\mathcal{H}_2$-PH w/order r=6



Combining POD and ensemble-free bases.





- POD is very accurate in capturing *observed* dynamics (with respect to a particular choice of input ensemble) — but not *unobserved yet feasible* dynamics.
- Enrich this POD basis by including components that are optimized for *arbitrary* (small) virtual inputs (e.g., the ensemble-free $\mathcal{H}_\infty/\mathcal{H}_2$-adapted bases).
- While the POD component brings in accurate approximations for inputs *similar* to the training ensemble, the linear optimal component may be expected to correctly adapt system behavior for as yet unobserved inputs.
- For *any* choice of reduction bases, the reduced system approximations remain structurally similar to the original system and in particular, will always be stable and passive.

Conclusions

- Reviewed basic notions of dissipative systems for LTI systems.
  - Key point: dissipativity is an exogenous property tied to a specific supply rate, not tied to a particular realization.
  - A particular realization gives rise to a family of storage functions (parameterized by solutions to an LMI).
- Introduced an interpolatory projection method that preserves dissipative system structure.
  - \+ Pro: Allows arbitrary state-space projection - gives potential for high-fidelity
  - − Con: Requires knowledge of a storage function

    (potentially intractable for large order)
- Nonlinear extensions for passive dynamical systems

  Port-Hamiltonian systems
  - ▷ Ensemble-based POD methods that preserve passivity.
  - ▷ Ensemble-free, asymptotically optimal methods that preserve passivity.
  - ▷ Combination of ensemble-based and ensemble-free bases.

Key point: turn an implicitly defined exogenous feature of the system (dissipativity) into an explicit structural feature for a realization that can be preserved with high fidelity model reduction strategies.